

# Star product and ordered moments of photon creation and annihilation operators

S N Filippov<sup>1</sup> and V I Man'ko<sup>1,2</sup>

<sup>1</sup> Moscow Institute of Physics and Technology, Dolgoprudnyi, Moscow Region, Russia

<sup>2</sup> P. N. Lebedev Physical Institute, Russian Academy of Sciences, Moscow, Russia

E-mail: [sergey.filippov@phystech.edu](mailto:sergey.filippov@phystech.edu), [manko@sci.lebedev.ru](mailto:manko@sci.lebedev.ru)

**Abstract.** We develop a star-product scheme of symbols defined by the normally ordered powers of the creation and annihilation photon operators,  $(\hat{a}^\dagger)^m \hat{a}^n$ . The corresponding phase space is a two-dimensional lattice with nodes  $(m, n)$  given by pairs of nonnegative integers. The star-product kernel of symbols on the lattice and intertwining kernels to other schemes are found in explicit form. Analysis of peculiar properties of the star-product kernel results in new sum relations for factorials. Advantages of the developed star-product scheme for describing dynamics of quantum systems are discussed and time evolution equations in terms of the ordered moments are derived.

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## 1. Introduction

This article concerns quantum states of one-mode radiation field and their time development by using normally ordered moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  of photon creation and annihilation operators,  $\hat{a}^\dagger$  and  $\hat{a}$ . Ordering of creation and annihilation operators was studied extensively in the 1960s in connection with the quasiprobability distributions (see, e.g., [1–3]). Thus, the Husimi-Kano function  $Q(\alpha)$  [4, 5] is closely related with the normally ordered density operator  $\hat{\rho} = \sum_{m,n} \rho_{mn}^{(n)} (\hat{a}^\dagger)^m \hat{a}^n$ , namely,  $Q(\alpha) = \sum_{m,n} \rho_{mn}^{(n)} (\alpha^*)^m \alpha^n$ , where  $\alpha^*$  denotes the complex conjugate of  $\alpha$ . Similarly, the Wigner  $W$ -quasidistribution [6] and the Sudarshan-Glauber  $P$ -quasidistribution [7, 8] are expressed through symmetrically and antinormally ordered representations of the density operator, respectively (see, e.g., [9, 10]). The quasidistributions on phase space do not limit to  $Q$ ,  $W$ , and  $P$ -functions and are reviewed in several papers (see, e.g., [10–12]). The recent detailed review of the phase-space approach is presented by Vourdas [13]. An advantage of the phase-space description of quantum states is that one deals with functions ( $\mathbb{C} \mapsto \mathbb{R}$ ) instead of operators. As equations for quasidistributions do not involve any operator, they are sometimes easier to handle than the Schrödinger or von Neumann equations [10]. Moreover, in the phase-space formalism, quantum phenomena are known to be interpreted in the classical-like manner [14].

Being a rather good alternative to the density operator, the quasidistribution functions  $Q$ ,  $W$ , and  $P$  are all equivalent in the sense that they all contain the thorough physically meaningful information about the state [10]. As the quasidistributions are equivalent, the question arises itself: why do not we use only one of them for all the problems? The answer lies in actual applications of the quasidistributions. For example, using the classical interpretation of the Wigner  $W$ -function in collision problems, it is possible to make some reasonable approximations and obtain an appropriate but still accurate solution to the problem without requiring excessive computer time and expense. On the other hand, the nonnegativity and smoothness of the Husimi-Kano  $Q$ -function make it advantageous for the analysis of classically chaotic nonlinear systems [10]. Also, the  $Q$ -function of the radiation field can be measured at optical frequencies via an eight-port homodyne detection scheme (see, e.g., [9]). Finally, the negativity of the Sudarshan-Glauber  $P$ -function is a commonly accepted criterion of the state's non-classicality, however, to observe the fact  $P(\alpha) < 0$  is not that easy (see, e.g., [15, 16]). We can draw a conclusion that although all the quasidistribution functions are equivalent, the different functions exhibit different properties and obey different dynamical equations. Those differences specify the most advantageous representation for a particular problem.

The set of normally ordered moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  can also be treated as a quasiprobability function of two nonnegative integers, i.e.  $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \mapsto \mathbb{C}$ . An interest to the ordered moments rose in the 1990s and was encouraged by the advances of measuring quantum states of light. Similar to the conventional quasidistributions, the moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  thoroughly determine the density operator  $\hat{\rho}$  and the explicit relation is found in the papers [17–21]. Despite the moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  are but one of

plenty equivalent prescriptions to deal with quantum states (see, e.g., [13]), there is a particular problem where the moments play an indispensable role: a measurement of itinerant microwave quantum states.

The matter is that at optical wavelengths there is no need to calculate the normally ordered moments and then use them to find the density operator. The output of the homodyne detection scheme is rotated quadrature distributions  $w(X, \theta)$ ,  $X \in \mathbb{R}$ ,  $\theta \in [0, \pi]$ , also referred to as optical tomogram (see, e.g., [22–24] and the review [25]). There exists an explicit formula for reconstructing the density operator  $\hat{\rho}$  in terms of the optical tomogram (see, e.g., [26–28]). The normally ordered moments are easily expressed through the measured optical tomogram as well [29, 30].

At microwave frequencies, conversely, to measure the rotated quadrature distributions  $w(X, \theta)$  is a challenge [31] because of the strong thermal noise added and the absence of single-photon detectors in such spectral region. However, it has been reported recently how to extract the lower-order moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  of microwave quantum states themselves and exclude the noise contribution to experimental data [32–34]. For details we refer the reader to the paper [35]. Thus, in the microwave domain, the lower-order moments are experimentally determined and contain the primary information about a quantum state.

Given only normally ordered moments of the microwave radiation field, it is reasonable to associate a quantum state with the measurable quasiprobability function  $f(m, n) \equiv \langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  of two nonnegative integers and consider the time evolution  $f(m, n; t)$  governed by a Hamiltonian  $\hat{H}$  in presence of decoherence processes. This motivates us to follow a star-product approach [36–42] and develop the particular star-product scheme on a two-dimensional lattice  $(m, n)$ , where any operator  $\hat{A}$  is associated with a symbol  $f_A(m, n) = \text{Tr}[(\hat{a}^\dagger)^m \hat{a}^n \hat{A}]$  and the product  $\hat{A}\hat{B}$  of two operators corresponds to a star product (also known as twisted product) of the corresponding symbols, i.e.  $f_{AB}(m, n) \equiv [f_A \star f_B](m, n)$ . Hence, one can associate the density operator  $\hat{\rho}$  of a quantum state with the normally ordered moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle \equiv f_\rho(m, n)$  and the Hamiltonian  $\hat{H}$  with its symbol  $\text{Tr}[(\hat{a}^\dagger)^m \hat{a}^n \hat{H}] \equiv f_H(m, n)$ . Then the unitary evolution of a quantum state can be easily written in terms of the introduced symbols as

$$\frac{\partial f_\rho(m, n)}{\partial t} = -\frac{i}{\hbar} [f_H \star f_\rho - f_\rho \star f_H](m, n), \quad m, n = 0, 1, \dots, \quad (1)$$

where  $\hbar$  is the Planck constant.

Formula (1) is nothing else but the evolution equation for the normally ordered moments. Therefore, the time development of a quantum state is expressed through the measurable characteristics, which resembles the expectation value approach [43, 44] and the tomographic-probability approach [28, 45, 46] to quantum mechanics. According to the latter one, the measurable tomographic probability is a primary object to describe quantum states. The evolution equations for tomograms were found, e.g., in [47–49].

A kernel that determines the star product  $\star$  in formula (1) is not known in the literature and is expressed through factorials in Section 3. Since the kernel of any star product scheme is to satisfy a nontrivial sum relation, we derive a new relation on

factorials in passing.

Let us note that a unitary evolution of any quasidistribution function can be written in the form of Eq. (1) (see, e.g., [10]) but the specificity of our particular case is that the symbol  $f_H(m, n)$  is not determined even for the harmonic oscillator Hamiltonian  $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$  because the trace is diverging. Nevertheless, in Section 4 we show that the symbols  $[f_H \star f_\rho](m, n)$  and  $[f_\rho \star f_H](m, n)$  can still be determined and Eq. (1) takes the form of a difference equation for  $f_\rho(m, n)$ .

The aim of this paper is to develop the star-product scheme of symbols given by the normally ordered moments, to explore properties of the star-product kernel, and to derive unknown evolution equations for the moments via the star-product approach. It is worth pointing out that we study the star-product scheme not of functions on the conventional phase space  $(q, p)$  but of functions on the two-dimensional lattice  $(m, n)$ , where  $m, n$  are nonnegative integers that determine the moments  $\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle$ . This peculiarity provides a nontrivial change of variables and, as we will see, yields the difference quantum basic equations in contrast to the partial differential equations for the quasidistributions  $Q(\alpha)$ ,  $W(\alpha)$ , and  $P(\alpha)$ .

The article is organized as follows.

In Section 2, we show how to treat operators as symbols defined by the normally ordered powers of creation and annihilation operators. Also, we revisit the reconstruction of operators given by their symbols, i.e. the problem of moments. In Section 3, we develop the star-product formalism [41, 42] of the introduced symbols. In particular, we find the star-product kernel and intertwining kernels. We calculate those kernels in the explicit form and use their particular properties to derive new sum relations involving factorials. In Section 4, we apply the developed star-product scheme to derive the time evolution of moments. Unitary and non-unitary evolutions are considered. Conclusions are presented in Section 5.

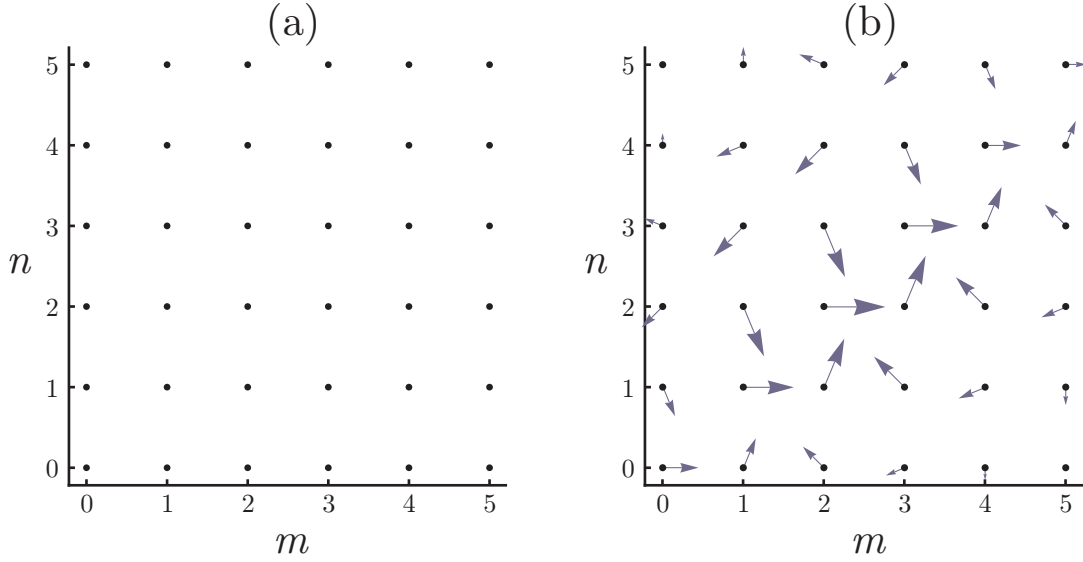
## 2. Quantization scheme: symbols of operators

Unless otherwise stated, we assume that the expression  $\text{Tr}[\cdot]$  is well defined. Thus, we intensionally avoid discussions of the convergence problems and focus our attention on the recipe to deal with symbols of operators.

Consider an operator  $\hat{A}$  acting on the same Hilbert space as the density operator  $\hat{\rho}$ . Assuming the existence of the trace

$$f_A(m, n) \equiv \text{Tr}[(\hat{a}^\dagger)^m \hat{a}^n \hat{A}] \equiv \text{Tr}[\hat{U}^\dagger(m, n) \hat{A}], \quad (2)$$

we will refer to the function  $f_A(m, n)$  of two discrete variables  $m, n = 0, 1, \dots$  as symbol of the operator  $\hat{A}$ . For the sake of convenience, we have introduced the dequantizer operator  $\hat{U}(m, n) \equiv (\hat{a}^\dagger)^n \hat{a}^m$ . Symbols  $f_A(m, n)$  are defined on the two-dimensional lattice  $(m, n)$  (Fig. 1a). An example of the symbol  $f_\rho(m, n)$  is illustrated for a squeezed vacuum state  $\rho$  in Fig. 1b.



**Figure 1.** (a) Phase space is a lattice whose nodes are labelled by two nonnegative integers  $(m, n)$ . Symbol  $f_\rho(m, n)$  is a complex valued function on the lattice. (b) Vectors  $(\text{Re}f_\rho(m, n)/m!n!; \text{Im}f_\rho(m, n)/m!n!)$  represent complex values of the symbol  $f_\rho(m, n)$  of the squeezed vacuum pure state  $\rho$  with squeezing parameter  $r = 1.3e^{i5\pi/4}$ .

The set  $\{f_A(m, n)\}_{m,n=0}^\infty$  is known to be informationally complete [17–21]. In other words, the operator  $\hat{A}$  can be reconstructed as follows:

$$\hat{A} = \sum_{m,n=0}^{\infty} f_A(m, n) \hat{D}(m, n), \quad (3)$$

$$\hat{D}(m, n) = \frac{1}{m!n!} \sum_{j=-\{m,n\}}^{\infty} \frac{(-1)^j (m+n+j)!}{(m+j)!(n+j)!} (\hat{a}^\dagger)^{n+j} \hat{a}^{m+j}, \quad (4)$$

where  $\{m, n\} = \min(m, n)$  and the operator  $\hat{D}(m, n)$  is called quantizer.

Using the Fock state representation of powers of creation and annihilation operators

$$(\hat{a}^\dagger)^m = \sum_{k=0}^{\infty} \sqrt{\frac{(m+k)!}{k!}} |m+k\rangle \langle k|, \quad \hat{a}^n = \sum_{k=0}^{\infty} \sqrt{\frac{(n+k)!}{k!}} |k\rangle \langle n+k|, \quad (5)$$

the dequantizer and quantizer can be rewritten in the form

$$\hat{U}(m, n) = \sum_{k=0}^{\infty} \frac{\sqrt{(n+k)!(m+k)!}}{k!} |n+k\rangle \langle m+k|, \quad (6)$$

$$\hat{D}(m, n) = \sum_{j=0}^{\{m,n\}} \frac{(-1)^j |n-j\rangle \langle m-j|}{j! \sqrt{(m-j)!(n-j)!}}. \quad (7)$$

The dequantizer and quantizer are shown to be orthogonal in the sense of trace operation [17–21]

$$\text{Tr}[\hat{U}^\dagger(m, n) \hat{D}(m', n')] = \delta_{m,m'} \delta_{n,n'}, \quad (8)$$

where  $\delta_{i,j}$  is the conventional Kronecker delta-symbol.

Let us now check that Eqs. (6)–(7) define an informationally complete scheme. It is shown in the papers [50–52] that a quantization scheme is informationally complete (tomographic) if and only if

$$\sum_{m,n=0}^{\infty} ||D(m,n)\rangle\rangle\langle\langle U(m,n)|| = \hat{\mathcal{I}}, \quad (9)$$

where  $\hat{\mathcal{I}}$  is the identity super-operator,  $||D(m,n)\rangle\rangle$  and  $||U(m,n)\rangle\rangle$  are vectors constructed from the quantizer and the dequantizer by the procedure of representing matrices as vectors. In our case we have

$$||U(m,n)\rangle\rangle = \sum_{k=0}^{\infty} \frac{\sqrt{(n+k)!(m+k)!}}{k!} |n+k\rangle \otimes |m+k\rangle, \quad (10)$$

$$||D(m,n)\rangle\rangle = \sum_{j=0}^{\{m,n\}} \frac{(-1)^j |n-j\rangle \otimes |m-j\rangle}{j! \sqrt{(m-j)!(n-j)!}}. \quad (11)$$

Direct calculation yields

$$\begin{aligned} & \langle p| \otimes \langle q| \left( \sum_{m,n=0}^{\infty} ||D(m,n)\rangle\rangle\langle\langle U(m,n)|| \right) |r\rangle \otimes |s\rangle \\ &= \sqrt{\frac{r!s!}{q!p!}} \delta_{p+s,q+r} \sum_{n=p}^r \frac{(-1)^{n-p}}{(r-n)!(n-p)!} \\ &= \sqrt{\frac{r!s!}{q!p!}} \delta_{p+s,q+r} \frac{(1-1)^{r-p}}{(r-p)!} = \sqrt{\frac{r!s!}{q!p!}} \delta_{p+s,q+r} \delta_{p,r} = \delta_{p,r} \delta_{q,s}, \end{aligned} \quad (12)$$

that is the relation (9) holds true indeed.

### 3. Star product

By definition, a star product of symbols  $f_A$  and  $f_B$  is nothing else but the symbol  $f_{AB}(m,n)$  of the product of two operators  $\hat{A}$  and  $\hat{B}$ , namely,

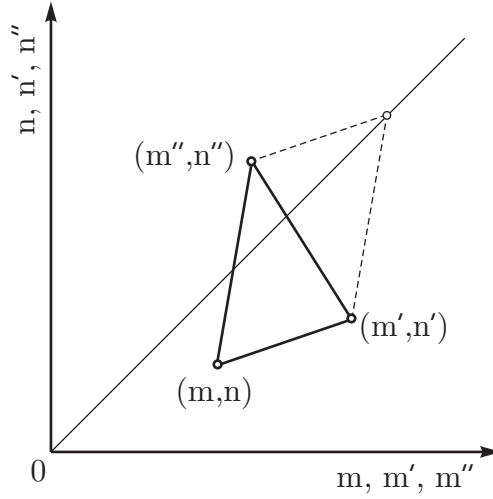
$$\begin{aligned} (f_A \star f_B)(m,n) &\equiv f_{AB}(m,n) \\ &= \sum_{m',n',m'',n''=0}^{\infty} f_A(m',n') f_B(m'',n'') K(m,n;m',n';m'',n''), \end{aligned} \quad (13)$$

where the kernel  $K$  is expressed in terms of the dequantizer and quantizer operators as follows:

$$K(m,n;m',n';m'',n'') = \text{Tr}[\hat{U}^\dagger(m,n) \hat{D}(m',n') \hat{D}(m'',n'')]. \quad (14)$$

From definition (13) it follows immediately that the star product is associative and the star-product kernel satisfies a nontrivial relation

$$\begin{aligned} & \sum_{k,l=0}^{\infty} K(m,n;k,l;m''',n''') K(k,l;m',n';m'',n'') \\ &= \sum_{k,l=0}^{\infty} K(m,n;m',n';k,l) K(k,l;m'',n'';m''',n'''), \end{aligned} \quad (15)$$



**Figure 2.** The kernel  $K(m, n; m', n'; m'', n'')$  is nonzero if the points  $(m, n)$ ,  $(m', n')$ , and  $(m'', n'')$  determine a parallelogram such that its forth vertex, that is opposite to  $(m, n)$ , belongs to a bisecting line. Also the conditions  $m \leq m''$ ,  $n \leq n'$  are to be met.

which is a consequence of the relation  $f_A \star f_B \star f_C = (f_A \star f_B) \star f_C = f_A \star (f_B \star f_C)$ .

Substituting (6) and (7) for the dequantizer and quantizer in (14), after some algebra, we obtain the star-product kernel in the explicit form

$$\begin{aligned} K(m, n; m', n'; m'', n'') &= \frac{(-1)^{m'-n''} (m' + m'' - m)!}{m'! n''! (m'' - m)! (n' - n)!} \delta_{m+n'+n'', n+m'+m''} \\ &= \frac{(-1)^{m'-n''} \Gamma(m' + m'' - m + 1) \delta_{m+n'+n'', n+m'+m''}}{\Gamma(m' + 1) \Gamma(n'' + 1) \Gamma(m'' - m + 1) \Gamma(n' - n + 1)}, \end{aligned} \quad (16)$$

where  $\Gamma$  is the conventional Euler gamma function.

Using the property of the Kronecker delta-symbol, the kernel can also be rewritten as follows:

$$K(m, n; m', n'; m'', n'') = F(m', m'' - m) F(n'', n' - n) \delta_{m+n'+n'', n+m'+m''}, \quad (17)$$

where  $F(a, b) = (-1)^a \sqrt{(a+b)!}/a!b!$ . In Fig. 2 we illustrate the restrictions on arguments of the kernel (16) under which it can take non-zero values.

Let us now consider how the nontrivial relation (15) looks like for the kernel (16). Substituting (17) for  $K$  in (15), we obtain the equality

$$\begin{aligned} &\sum_k F(k, m''' - m) F(m', m'' - k) F(n''', k + m''' - m - n''') F(n'', m' + m'' - n'' - k) \\ &= \sum_l F(m', l + n' - n - m') F(l, n' - n) F(m'', n'' + n''' - m'' - l) F(n''', n'' - l) \end{aligned} \quad (18)$$

which is to hold true whenever  $m' + m'' + m''' + n = n' + n'' + n''' + m$ . Expressing  $n'$  through the other variables and using the explicit form of the function  $F$  yields

$$\sum_{l=[0, n''+n'''-m'-m'']^{\{n''+n'''-m'', n''\}}} \frac{(-1)^l (l + m' + m'' + m''' - n'' - n''' - m)! (n'' + n''' - l)!}{l! (l + m'' + m''' - n'' - n''' - m)! (n'' + n''' - m'' - l)! (n'' - l)!}$$

$$\begin{aligned}
 &= \frac{(-1)^{n''+m''} m''! (m' + m'' + m''' - n'' - n''' - m)!}{(m''' - m)! n''!} \\
 &\times \sum_{k=\lceil 0, m+n'''-m''' \rceil}^{\{m'', m'+m''-n''\}} \frac{(-1)^k (k + m''' - m)! (m' + m'' - k)!}{k! (k - m - n''' - m''')! (m'' - k)! (m' + m'' - n'' - k)!}, \quad (19)
 \end{aligned}$$

where  $[a, b] = \max(a, b)$  and  $\{a, b\} = \min(a, b)$ . Although the obtained relation is rather complicated, it is valid for all non-negative integers  $m, m', m'', m''', n, n'', n'''$ . For instance, if we put  $m' + m'' - n'' = 0$ , then the summation over  $k$  in the right-hand side of Eq. (19) reduces to a single term  $k = 0$  provided  $m + n''' - m''' \leq 0$ . As a result we derive a new property

$$\sum_{l=\lceil 0, m'-M \rceil}^{\{n'''+m', n''\}} \frac{(-1)^l (M + l)! (n'' + n''' - l)!}{l! (M - m' + l)! (n''' + m' - l)! (n'' - l)!} = (-1)^{m'}, \quad (20)$$

where  $M = m''' - m - n''' \geq 0$  and  $n'' \geq m'$ . Note that the result of summation does not depend on  $m'', n'', n'''$ . In particular, if we choose  $n''' = 0$ , then

$$\sum_{l=\lceil 0, m'-M \rceil}^{m'} \frac{(-1)^l (M + l)!}{l! (M - m' + l)! (m' - l)!} = (-1)^{m'}. \quad (21)$$

Similarly, if we fix  $m' = 0$  in Eq. (20), then

$$\sum_{l=0}^{\{n'', n'''\}} \frac{(-1)^l (n'' + n''' - l)!}{l! (n'' - l)! (n''' - l)!} = 1. \quad (22)$$

### 3.1. Intertwining kernel between star-product schemes

Along with the considered star-product scheme with dequantizers (6) and quantizers (7), there exist many other star-product schemes. In order to distinguish them let us use superscripts  $(\mathfrak{N})$  for the just developed scheme of the normally ordered moments and write  $\hat{U}^{(\mathfrak{N})}(m, n)$  and  $\hat{D}^{(\mathfrak{N})}(m, n)$ .

As an example of the other quantization on the lattice  $(m, n)$  we may consider the star-product scheme with identical dequantizers and quantizers  $\hat{U}^{(\mathfrak{F})}(m', n') = \hat{D}^{(\mathfrak{F})}(m', n') = |m'\rangle\langle n'|$ . The symbols, that correspond to the two different schemes, are related by virtue of formulas

$$f_A^{(\mathfrak{N})}(m, n) = \sum_{m', n'=0}^{\infty} K_{\mathfrak{F} \rightarrow \mathfrak{N}}(m, n; m', n') f_A^{(\mathfrak{F})}(m', n'), \quad (23)$$

$$f_A^{(\mathfrak{F})}(m', n') = \sum_{m, n=0}^{\infty} K_{\mathfrak{N} \rightarrow \mathfrak{F}}(m', n'; m, n) f_A^{(\mathfrak{N})}(m, n), \quad (24)$$

where the intertwining kernels are expressed through dequantizers and quantizers as follows:

$$K_{\mathfrak{F} \rightarrow \mathfrak{N}}(m, n; m', n') = \text{Tr}[\hat{U}^{(\mathfrak{N})\dagger}(m, n) \hat{D}^{(\mathfrak{F})}(m', n')] = \frac{\sqrt{m'!n'!}}{(n' - m)!} \delta_{m+m', n+n'}, \quad (25)$$

$$K_{\mathfrak{N} \rightarrow \mathfrak{F}}(m', n'; m, n) = \text{Tr}[\hat{U}^{(\mathfrak{F})\dagger}(m', n') \hat{D}^{(\mathfrak{N})}(m, n)] = \frac{(-1)^{m-n'} \delta_{m+m', n+n'}}{(m - n')! \sqrt{m'!n'!}}. \quad (26)$$



It is worth mentioning that the dequantizers and quantizers of a star-product scheme should not necessarily depend on discrete variables. For example, in optical tomography, the dequantizer is a projection on the rotated quadrature, i.e.  $\hat{U}^{(\mathcal{D})}(X, \theta) = |X, \theta\rangle\langle X, \theta|$ , where  $(\hat{q} \cos \theta + \hat{p} \sin \theta)|X, \theta\rangle = X|X, \theta\rangle$ ,  $\hat{q}$  and  $\hat{p}$  are the position and momentum operator, respectively. The intertwining formulas that connect the optical tomogram and the normally (antinormally) ordered moments are derived in the papers [29, 30, 35].

#### 4. Evolution equations

As it is outlined in Section 1, the star-product formalism is quite useful for describing evolution of quantum states. In case of the star-product scheme (2)–(4), one deals with the functions of discrete variables instead of operators. The more important fact is that the functions  $f_\rho(m, n; t)$  are experimentally measurable [32–34] at different time moments  $t$ , giving an opportunity to observe dynamics of the system and motivating us to derive the evolution equations in terms of measurable quantities.

To start with, the time dependent and stationary von Neumann equations for the density operator  $\hat{\rho}$  take the following form within the star-product formalism:

$$\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}] \Leftrightarrow \frac{\partial f_\rho(m, n; t)}{\partial t} = -i(f_H \star f_\rho - f_\rho \star f_H)(m, n; t), \quad (27)$$

$$\frac{1}{2}(\hat{H}\hat{\rho}_E + \hat{\rho}_E\hat{H}) = E\hat{\rho}_E \Leftrightarrow \frac{1}{2}(f_H \star f_{\rho_E} + f_{\rho_E} \star f_H)(m, n) = Ef_{\rho_E}(m, n), \quad (28)$$

where  $\hat{H}$  and  $E$  are the Hamiltonian and the permitted energy level, respectively; the Planck constant  $\hbar = 1$ .

##### 4.1. Moments' dynamics for harmonic oscillator

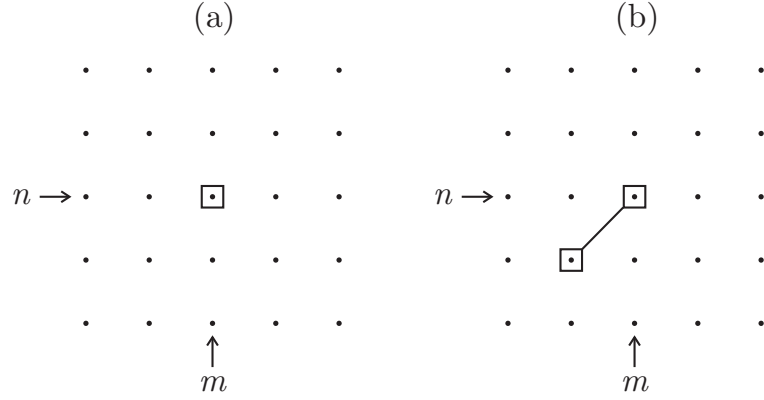
Let us consider a free evolution of the electromagnetic field governed by a harmonic oscillator Hamiltonian  $\hat{H} = \hat{a}^\dagger \hat{a} + \frac{1}{2}$ , where we have also used dimensionless units for the frequency ( $\omega = 1$ ). In this case we immediately encounter a problem of finding the symbol  $f_H(m, n)$  because it takes infinite values if  $m = n$ . However, this difficulty can be avoided if we focus on the star product of symbols. In fact,

$$\begin{aligned} (f_H \star f_\rho)(m, n) &= \text{Tr}[(\hat{a}^\dagger)^m \hat{a}^n (\hat{a}^\dagger \hat{a} + \tfrac{1}{2}) \hat{\rho}] = \text{Tr}[(\hat{a}^\dagger)^m (\hat{a}^n \hat{a}^\dagger) \hat{a} \hat{\rho}] + \tfrac{1}{2} \text{Tr}[(\hat{a}^\dagger)^m \hat{a}^n \hat{\rho}] \\ &= \text{Tr}[(\hat{a}^\dagger)^m (\hat{a}^\dagger \hat{a}^n + n \hat{a}^{n-1}) \hat{a} \hat{\rho}] + \tfrac{1}{2} \text{Tr}[(\hat{a}^\dagger)^m \hat{a}^n \hat{\rho}] \\ &= f_\rho(m+1, n+1) + (n + \tfrac{1}{2}) f_\rho(m, n). \end{aligned} \quad (29)$$

Similarly, we obtain  $(f_\rho \star f_H)(m, n) = f_\rho(m+1, n+1) + (m + \frac{1}{2}) f_\rho(m, n)$ . Substituting these results in Eqs. (27)–(28) yields

$$\frac{\partial f_\rho(m, n; t)}{\partial t} = i(m - n) f_\rho(m, n; t), \quad (30)$$

$$f_{\rho_E}(m+1, n+1) + \frac{m+n+1}{2} f_{\rho_E}(m, n) = E f_{\rho_E}(m, n). \quad (31)$$



**Figure 3.** Local dynamics of the moment  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  on the “phase-space” lattice: (a) harmonic oscillator and (b) damped harmonic oscillator. Squares denote nodes for which the corresponding moments are involved in the evolution of  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$ .

It is not hard to check that symbols  $f_{|N\rangle\langle N|}(m, n; t) = N!e^{i(m-n)t}\delta_{m,n}/(N-m)!$  of the Fock states  $|N\rangle$  such that  $m = n \leq N$  do satisfy the derived equations (30)–(31) if  $E = N + \frac{1}{2}$ .

In case of the harmonic oscillator Hamiltonian, the dynamics (30) of quantum states on the lattice  $(m, n)$  reduces to  $f_\rho(m, n; t) = f_\rho(m, n; 0)e^{i(m-n)t}$ , i.e. the moments simply gain phases in accordance with their position on the lattice. The corresponding vectors in Fig. 1b save their length and rotate with frequencies  $\omega_{mn} = m - n$ , so in time interval  $t = 2\pi$  all the vectors come back to the initial position. Thus, evolution of the moment  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  is extremely local on the “phase-space” lattice and does not depend on the values of moments in surrounding nodes (Fig. 3a).

#### 4.2. Moments’ dynamics for damped harmonic oscillator

Let us consider a damped evolution equation for the density operator described by the usual master equation (see, e.g., [53, 54])

$$\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{a}^\dagger \hat{a}, \hat{\rho}] + \gamma(1 + \nu)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger \hat{a}) + \gamma\nu(2\hat{a}^\dagger \hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger \hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \quad (32)$$

where  $\gamma$  is the damping coefficient and  $\nu$  is the equilibrium mean number of photons in a given mode.

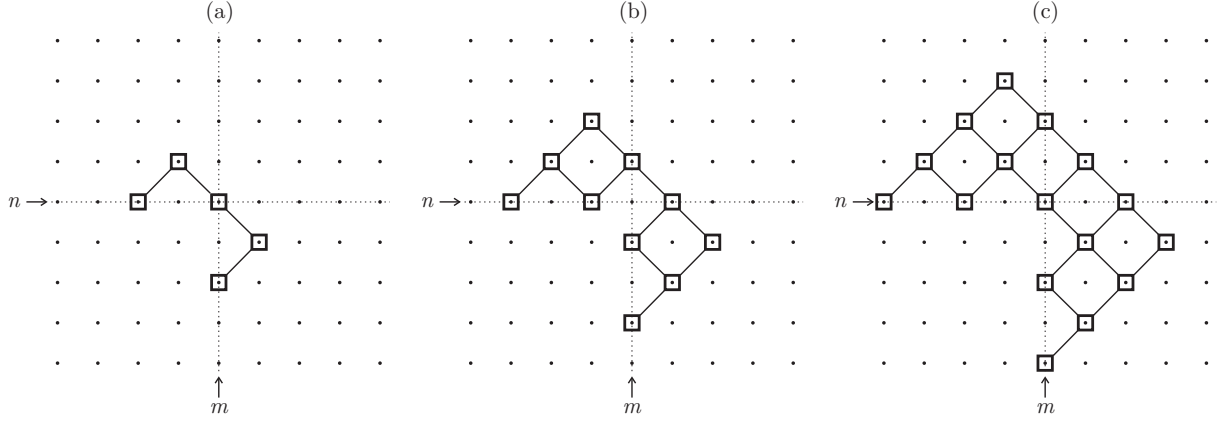
Arguing as above, we find the star product of symbols in question

$$(f_a \star f_\rho \star f_{a^\dagger})(m, n) = f_\rho(m+1, n+1), \quad (33)$$

$$\begin{aligned} (f_{a^\dagger} \star f_\rho \star f_a)(m, n) &= f_\rho(m+1, n+1) + (m+n+1)f_\rho(m, n) \\ &\quad + mn f_\rho(m-1, n-1). \end{aligned} \quad (34)$$

Now one can rewrite (32) in terms of the measurable moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle \equiv f_\rho(m, n)$  as follows:

$$\frac{\partial f_\rho(m, n; t)}{\partial t} = [i(m-n) - \gamma(m+n)] f_\rho(m, n; t) + 2\gamma\nu mn f_\rho(m-1, n-1; t). \quad (35)$$



**Figure 4.** Nodes on the “phase-space” lattice for which the corresponding moments are involved in the evolution of a particle: (a) kinetic term  $\hat{p}^2/2$ ,  $V(\hat{q}) \propto \hat{q}^2$ ; (b)  $V(\hat{q}) \propto \hat{q}^3$ ; (c)  $V(\hat{q}) \propto \hat{q}^4$ . The lower sum  $m + n$  for the marked nodes the greater weight of corresponding moments in the time evolution.

In case of the damped harmonic oscillator, the dynamics of moments is again quite local on the lattice  $(m, n)$  and involves only two nodes (Fig. 3b). It is worth mentioning, that Eq. (35) could also be derived by using a connection between the normally ordered moments and the Wigner function, and then substituting these relations in the Fokker-Planck equation for the Wigner function [35]. However, as we can see, the star-product approach is straightforward to derive evolution equations for measurable quantities like the normally ordered moments of the creation and annihilation photon operators.

#### 4.3. Moments' dynamics for a particle

Although the background of our consideration is measuring radiation fields at microwaves, the developed star-product formalism can be also applied to the one-dimensional motion of particles governed by the Hamiltonian  $\hat{H} = \hat{p}^2/2 + V(\hat{q}) = -(\hat{a} - \hat{a}^\dagger)^2/4 + V((\hat{a} + \hat{a}^\dagger)/\sqrt{2})$ . The main point is that the time evolution of moments remains local on the lattice if the potential energy  $V(q)$  can be approximated by lower-order terms of the Taylor expansion (Fig. 4).

If  $V(q) = q^l$ , then the nodes  $(m', n')$  involved in the dynamics of the moment  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  form a truncated square with center at the point  $(m - 1, n - 1)$ . The nodes  $(m', n')$  satisfy the relations

$$|m' - m + 1| + |n' - n + 1| = \begin{cases} 0, 2, 4, \dots, l & \text{for even } l, \\ 1, 3, 5, \dots, l & \text{for odd } l, \end{cases}$$

$$\max(m' - m, n' - n) \geq 0.$$
(36)

It is instructive to compare the  $W$ -function evolution and the evolution of moments. The derivatives  $d^l V/dq^l$  and powers  $q^l$  form the Taylor expansion of  $V(q)$ . These

derivatives show which partial derivatives  $\partial^l W / \partial p^l$  contribute to the evolution on the conventional phase space  $(q, p)$ , whereas the powers  $q^l$  show which moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$  contribute to the evolution on the “phase-space” lattice  $(m, n)$ . The important practical difference between a quasidistribution on the conventional phase space  $(q, p)$  and our proposal is that the time development of a conventional quasidistribution is given in terms of the infinite-order partial differential equation which is not so easy to solve. A numerical solution of such an equation would require a construction of a two-dimensional rectangular grid in the  $(q, p)$ -plane, with the size and density of the grid being taken according to a desired accuracy (see, e.g., [10]). The partial derivatives are then replaced by finite differences. The higher the order of the derivative, the more nodes of the grid are involved. Dealing with the ordered moments, one does not need to introduce any artificial grid because of the lattice itself. The time development is then given by the exact difference equations in contrast to the approximate finite-difference equations for quasidistributions on the  $(q, p)$ -plane.

## 5. Conclusions

To conclude, we present the main results of the paper.

An analysis of the star-product scheme based on the normally ordered creation and annihilation photon operators has been motivated by the recent advances in measuring ordered moments for microwave quantum states [32–34]. In addition, the phase space of the constructed quantization scheme is a two-dimensional lattice  $(m, n)$  whose nodes are given by two nonnegative integers. Such a structure of the phase space is advantageous for describing the time development of some quantum systems because the exact evolution equations take the form of difference equations in contrast to the partial differential equations for conventional quasidistributions on the  $(q, p)$ -plane usually approximated by finite-difference equations on the  $(q_i, p_j)$ -grid of rather artificial size and density.

Moreover, it is quite reasonable to define a quantum state evolution in terms of the measurable quantities, so we have filled a gap of such equations in terms of the normally ordered moments  $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$ . The dynamics of moments is shown to be local on the lattice for (damped) radiation fields and particles moving in smooth potentials.

Another substantial result is that the star-product kernel is found in the explicit form (16). As any star-product kernel is to satisfy specific non-linear equalities, we have applied one of those equalities to the obtained kernel and thus derived new sum relations involving factorials (19)–(22).

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